EFFECT OF A NONHOLONOMIC CONSTRAINT ON THE STABILIZABILITY OF A MECHANICAL SYSTEM

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Change in stabilizability of a mechanical system under the influence of a nonholonomic constraint is studied, and conditions given, under which the imposition of such a constraint leads to impairment of the stabilizability of the system. Some cases are shown in which the imposition of a nonholonomic constraint, however, improves the stability of the system.

1. For the sake of completeness, we shall begin by quoting some well-known facts from the theory of stabilization of nonholonomic systems [1 – 4].

Let a mechanical system be given whose position is defined in the terms of generalized coordinates $q_1,\ldots, q_{n+1}$ and let the motion of the system be subject to a nonholonomic constraint whose equation can be written in the form

$$q_{n+1}^{\cdot} = \sum_{i=1}^{n} \omega_i (q_1, \ldots, q_{n+1}) q_i^{\cdot} \tag{1.1}$$

We assume that the system in question has a one-parameter manifold Q of unstable equilibrium positions [2]

$$q_i = q_i(q), \ \alpha \leqslant q \leqslant \beta \ (\alpha, \ \beta = \text{const}), \ (i = 1,\ldots, n+1) \tag{1.2}$$

We shall consider the problem [1 and 3] of determining a control $u(q_1,\ldots, q_{n+1}, \dot q_1, \ldots, \dot q_{n+1})$, which would cause the equilibrium positions (1.2) to become asymptotically stable. Following the accepted procedure, we shall start by setting up the equations of perturbed motion with the help of new coordinates $s_1,\ldots, s_n$ and ξ given by

$$s_i = q_i - q_i(q) \qquad (i = 1,\ldots, n+1)$$

$$\xi = s_{n+1} - \sum_{i=1}^{n} \omega_i (q_1(q), \ldots, q_{n+1}(q)) s_i$$

where ω_i denote the coefficients appearing in Eq. (1.1) of the nonholonomic constraint. Further, using the equations of motion of a nonholonomic system [5 and 6] in the form due to Appell or Lagrange, we obtain the equations of perturbed motion in the form [4 and 7]

$$\dot z = A(q) z + b(q)u + c(q)\xi + \varphi_1(q, \xi, z, u)$$

$$\dot\xi = \varphi_2(q, \xi, z) \qquad (\varphi_2(q, \xi, 0) \equiv 0) \tag{1.3}$$

Here $z = \{z_1,\ldots, z_{2n}\}$ denotes a $2n$-dimensional vector whose components are given by $z_{2i-1} = s_i$, $z_{2i} = s_i^{\cdot}$ $(i = 1,\ldots, n)$; $A(q)$ is a $2n \times 2n$-matrix, while $b(q)$, $c(q)$ and $\varphi_1(q, \xi, z, u)$ are $2n$-dimensional vectors. We shall assume that the elements of the matrix A, vectors b, c and φ_1 and of the functions φ_2 are all analytic in q, ξ, z_i and u

and that the expansions of φ_1 and φ_2 begin with the terms which contain second order infinitesimals in ξ, z_i and u.

From (1.3) it follows that we are dealing with the critical case of a single zero root [1], therefore we conclude that the control cannot stabilize the equilibrium positions (1.2) up to the asymptotic stability in the Liapunov sense. Consequently, we adopt the following definition.

Definition 1.1. We shall call the manifold of equilibrium positions (1.2) of a nonholonomic system asymptotically stable, if any of these states is stable in the Liapunov sense and, if for any perturbed motion $q_i(q^*)$ adjacent to any of these states the condition

$$\lim_{t\to\infty} q_i(t) = q_i(q^* + \varepsilon) \quad (i = 1,\ldots, n+1) \quad (\alpha \leqslant q^* \leqslant \beta)$$

holds. Here ε denotes a sufficiently small number, provided that the initial perturbations $q_i - q_i(q^*)$ $(i = 1,\ldots, n+1)$ are themselves small. In this connection we shall next consider a problem of stabilizing the manifold of equilibrium states (1.2) of a nonholonomic system.

Problem 1.1. To find a control $u = u(q,z)$ such, that the manifold of the equilibrium positions (1.2) becomes asymptotically stable in the sense of Definition 1.1.

The following theorem then holds [4].

Theorem 1.1. If the linear system

$$\dot z = A(q)z + b(q)u \tag{1.4}$$

becomes asymptotically stable in the Liapunov sense under the action of u for any q belonging to the segment $[\alpha, \beta]$, then the Problem 1.1 has a solution, i.e. the system (1.3) can be stabilized in the sense of the Definition 1.1 and the stabilizing control u has the form

$$u(q, z) = \sum_{i=1}^{2n} p_i(q)\, z_i$$

where $p_i(q)$ are analytic in q.

2. We shall now consider a problem of stabilizing a system consisting of two heavy material points M_1 and M_2 connected with a thin, weightless rod in such a manner, that M_1 is fixed rigidly to the rod while M_2 is free to slide along the rod without friction. Let the points M_1 and M_2 be situated on the surface of two ellipsoids. Then the system will have an unstable position of equilibrium corresponding to the case when both points, M_1 and M_2, are positioned at the peaks of these ellipsoids. We shall assume that the set of points M_1 and M_2 is subject to internal attractive (or repulsive) forces and as generalized coordinates we shall use (x_1, y_1) for M_1 and (x_2, y_2) for M_2, both belonging to the Cartesian coordinate system whose z axis is directed upwards in the direction which is vertical at the given part of the Earth globe. Let the kinetic and potential energy of the system be given by

$$T = {}^1/_2 (x_1^{\cdot 2} + y_1^{\cdot 2} + x_2^{\cdot 2} + y_2^{\cdot 2}) + T_1(x, y, x^\cdot, y^\cdot)$$

$$\Pi = -{}^1/_2 [{}^1/_2 (x_1 + y_1 - 1)^2 + 2(y_1 - x_1 + 1)^2 + x_2^2 + 6x_2y_2 + 6y_2^2] + \Pi_1(x, y)$$

We assume that the functions T_1 and Π_1 can be expanded into series in some neighborhood of the equilibrium point

$$x_1 = 1,\ y_1 = x_2 = y_2 = 0 \tag{2.1}$$

and, that the expansion will begin with the terms of order not lower than the third in x_i, y_i, $x_i^\cdot$ and $y_i^\cdot$. Then the equations of motion of the system M_1 and M_2 will be given, in the linear approximation, by

$$z_1'' = {}^5/_2 z_1 - {}^3/_2 z_2 - u, \quad z_2'' = -{}^3/_2 z_1 + {}^5/_2 z_2, \quad z_3'' = z_3 + 3z_4 + u, \quad z_4'' = 3z_3 + 6z_4 \tag{2.2}$$

where z_i denote the deviations from the equilibrium position (2.1), i.e.

$$z_1 = x_1 - 1, \quad z_2 = y_1, \quad z_3 = x_2, \quad z_4 = y_2$$

Constructing the matrix A_1 and the vector b_1

$$A_1 = \begin{Vmatrix} 2.5 & -1.5 & 0 & 0 \\ -1.5 & 2.5 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 6 \end{Vmatrix}, \qquad b_1 = \begin{Vmatrix} -1 \\ 0 \\ 1 \\ 0 \end{Vmatrix}$$

we find, that the vectors $b_1, A_1 b_1, A_1^2 b_1$ and $A_1^3 b_1$ are linearly independent, which implies that the system (2.2) can be stabilized [1, 3 and 8] by the control u to the asymptotic stability in the Liapunov sence. This fact could, however, also be deduced from the general considerations of stabilization of the holonomic systems [8], since in the present case the straight line connecting the points M_1 and M_2 in their position of equilibrium does not coincide with any of the principal directions of the potential energy surfaces $\Pi_1^{(i)} = \text{const}$ $(i = 1, 2)$ of the points M_1 and M_2, respectively.

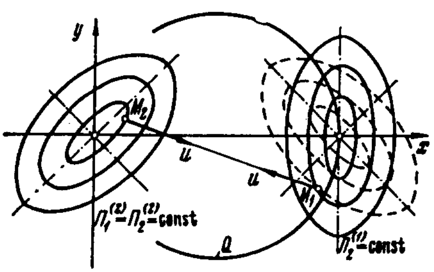

Fig. 1

We note that the imposition of a nonholonomic constraint of the form (1.1) on the system (2.2) can, by the theorem 1.1, only impair its stabilizability.

Let us now rotate the ellipse $\Pi_1^{(1)} = \text{const}$ by 45° clockwise. Then the line connecting the points M_1 and M_2 in their equilibrium position, will coincide with the principal axis of the ellipse $\Pi_2^{(1)} = \text{const}$ (see Fig. 1) and the resulting holonomic system will no longer be stabilizable. Indeed, we shall then have

$$\Pi = -{}^1/_2 \left[(x_1 - 1)^2 + 4y_1^2 + x_2^2 + 6x_2 y_2 + 6y_2^2\right] + \Pi_2(x, y) \tag{2.3}$$

The set of points M_1 and M_2 will, obviously, still have the position of equilibrium (2.1). Its motion in the neighborhood of this position will now be defined by the following first approximation equations

$$z_1'' = z_1 - u, \quad z_2'' = 4z_2, \quad z_3'' = z_3 + 3z_4 + u, \quad z_4'' = 3z_3 + 6z_4 \tag{2.4}$$

As expected [1 and 8], the system (2.4) cannot be stabilzed by the force u directed along the rod.

We shall show that the stabilizability of the system (2.4) can be improved by imposing a nonholonomic constraint on it. To do this, we shall fit the point M_1 with a wheel possessing a sharp edge, sp that the velocity of M_1 is, at all times, directed along the rod. In other words, we shall impose on the system M_1 and M_2 a nonholonomic constraint of the form

$$(y_2 - y_1)\, x_1' - (x_2 - x_1)\, y_1' = 0 \tag{2.5}$$

Since the system (2.4) is assumed to be the initial one, we suppose, as before, that the potential energy of the system is given by (2.3). After the necessary computations [2 and 7] we find, that the set of points M_1 and M_2 has a one-parameter manifold Q of the positions of equilibrium

$$x_1 = q,\ y_1 = \pm {}^1/_2\sqrt{q(1-q)}\ (0 < q \leqslant 1),\ x_2 = y_2 = 0 \tag{2.6}$$

The value $q = 1$ corresponds to the equilibrium position (2.1) in which we are interested. When $q = 1$, the system (1.4) characterizing the stabilizability of a nonholonomic system has the form

$$\begin{aligned} z_1^{\cdot\cdot} &= z_1 - u & (z_1 &= x_1 - 1) \\ z_2^{\cdot\cdot} &= z_2 + 3z_3 + u & (z_2 &= x_2) \\ z_3^{\cdot\cdot} &= 3z_2 + 6z_3 & (z_3 &= y_2) \end{aligned} \tag{2.7}$$

System (2.7) becomes, under the action of u, asymptotically stable in the Liapunov sence. This follows from the fact that the vectors b_2, A_2b_2 and A_2b_2 where

$$A_2 = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 6 \end{Vmatrix} \qquad b_2 = \begin{Vmatrix} -1 \\ 1 \\ 0 \end{Vmatrix}$$

are linearly independent. Consequently the system (1.4) can also be stabilized when $0 < \alpha \leqslant q \leqslant 1$. Theorem 1.1 therefore, implies that the nonstabilizable holonomic system M_1 and M_2 can now be stabilized in the sense of the definition 1.1 provided that the nonholonomic constraint (2.5) is imposed on it.

Since the example was chosen only to illustrate the method, we used the simplest model of a mechanical system consisting of two points. The argument however, can easily be extended to the general case.

Let the holonomic mechanical system be controlled by the force u, whose direction does not coincide with any if the directions of the principal normal coordinates. Such a system can be stabilized asymptotically in the Liapunov's sense [8]. Consequently, a nonholonomic constraint can, by Theorem 1.1, impair its stabilizability. Conversely, if the direction of the controlling force u coincides with any of the principal normal coordinates, then the system cannot be stabilzed. However, a nonholonomic constraint of the type (1.1) imposed on this system, restricts its displacements in the directions different from that of the normal coordinate coincident with u, and the system may then become asymptotically stable in the sense of the definition 1.1.

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